Letters

Approximations for the Symmetrical Parallel-Strip Transmission Line

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Abstract—A single approximation that is valid for the capacitance of both "wide" and "narrow" parallel-strip transmission lines was derived by assuming that the current distribution is uniform. An approximate formula for the repulsive force between the strips was also derived.

INTRODUCTION

Analysis of the finite-width parallel-strip transmission line diagramed in Fig. 1 is a challenging problem with a rich history of contributions by many notable workers, including Maxwell and J. J. Thomson. For a historical bibliography, the reader is referred to the thesis and paper by Black and Higgins [1].

The exact electrostatic solution for a homogeneous lossless dielectric can be obtained with a Schwarz–Christoffel transformation which leads to a free-space line capacitance of

$$C = \frac{\epsilon_0 K'}{K}\ (F/m) \tag{1}$$

where K and K' are complete elliptic integrals of the first kind. Unfortunately, this solution is extremely cumbersome to use because the elliptic-integral argument is not an explicit function of the line-shape ratio, $R = w/b$. The usefulness of the exact solution is also limited by the lack of elliptic-integral tabulations for arguments between 89 and 90°. For this reason, (1) cannot be conveniently evaluated for shape ratios $\gtrsim 2$.

Because of these difficulties, many approximate formulas for the capacitance and characteristic impedance have been developed. As far as is known, none of these is individually valid for both narrow and wide lines, which is not surprising, considering the large field variations experienced over the $R \ll 1$ to $R \gg 1$ shape-ratio range. The purpose of this communication is to show that an approximate formula that is reasonably accurate over the entire R range does exist, and that the formula can be derived by assuming that the current distribution is uniform across the conductors.

THE UNIFORM-CURRENT APPROXIMATION

Referring to Fig. 1, if the strips carry equal and opposite uniformly distributed currents I, then the repulsive force per unit length experienced by the filamentary current in a strip of width dz is [2]

$$dF(y) = \frac{\mu I^2 y\,dz}{2\pi w^2} \int_{-w/2}^{w/2} \frac{dx}{y^2 + (z-x)^2} \tag{2}$$

which integrates to

$$dF(y) = \frac{\mu I^2}{2\pi w^2}\left[\tan^{-1}\left(\frac{z + w/2}{y}\right) - \tan^{-1}\left(\frac{z - w/2}{y}\right)\right] dz. \tag{3}$$

Integrating (3) over $(-w/2 \le z \le w/2)$ gives

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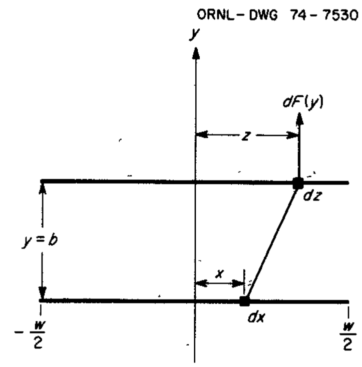

Fig. 1. Transmission line formed of finite-width parallel strips.

$$F(y) = \frac{\mu I^2}{\pi}\{(1/w)\tan^{-1}(w/y) - (y/2w^2)\ln[1 + (w/y)^2]\} \tag{4}$$

for the total repulsive force per unit length.

When the strips are separated by a distance b, the stored energy is

$$U(b) = -\int_0^b F(y)\,dy \tag{5}$$

if a repulsive force is assumed positive. Substitution of $u = w/y$ changes (5) to

$$U(R) = \int_\infty^R \frac{y^2}{w} F(u)\,du. \tag{6}$$

Substitution of (4) into (6) gives

$$U(R) = (\mu I^2/2\pi)\int_\infty^R [u^{-3}\ln(1+u^2) - 2u^{-2}\tan^{-1} u]\,du \tag{7}$$

which can be integrated by standard techniques to obtain

$$U(R) = \frac{\mu I^2}{\pi}\left[(1/R)\tan^{-1} R - (1/4R^2)\ln(1+R^2) + (1/4)\ln\left(\frac{1+R^2}{R^2}\right)\right] \tag{8}$$

for the energy stored per unit length.

Making (8) equal to $LI^2/2$ gives

$$L = \frac{2\mu}{\pi}\left[(1/R)\tan^{-1} R - (1/4R^2)\ln 1 + R^2) + (1/4)\ln\left(\frac{1+R^2}{R^2}\right)\right] \tag{9}$$

for the approximate inductance per unit length (H/m). If the parallel strips are assumed to form a loss-free transmission line, then the free-space capacitance per unit length can be computed from $LC = \mu_0\epsilon_0$ which gives

$$C = \left[(8/R)\tan^{-1} R - (2/R^2)\ln(1+R^2) + 2\ln\left(\frac{1+R^2}{R^2}\right)\right]^{-1}(4\pi\epsilon_0)\ \text{F/m}. \tag{10}$$

TABLE I
Approximate Values of $C/4\pi\epsilon_0$ for Parallel-Strip Transmission Lines

R	Eq (10)	Eq (11)	Eq. (12)	Eq (13)	Eq (14)
0.01	0 0410	0.0417	0 0417		
0 02	0 0462	0 0472	0 0472		
0.05	0 0556	0 0571	0 0571		
0.07	0.0601	0.0618	0 0618		
0.1	0.0657	0 0678	0 0678	0 0215	0 0803
0.2	0 0803	0 0835	0 0833	0 0470	0 0907
0 5	0.113	0 120	0 118	0 0941	0 121
0.7	0 132	0 143	0 139	0 119	0 140
1.0	0.159	0 180	0 165	0 152	0 168
2.0	0.246	0 361	0 210	0 249	0 258
5 0	0 496		0 086	0 511	0 515
7 0	0.661		0 045	0 678	0 681
10 0	0.905			0 926	0 928
20.0	1 715			1 739	1 740
50 0	4.119			4 150	4.150
70.0	5 701			5 750	5.750
100.0	8 117			8 146	8 146

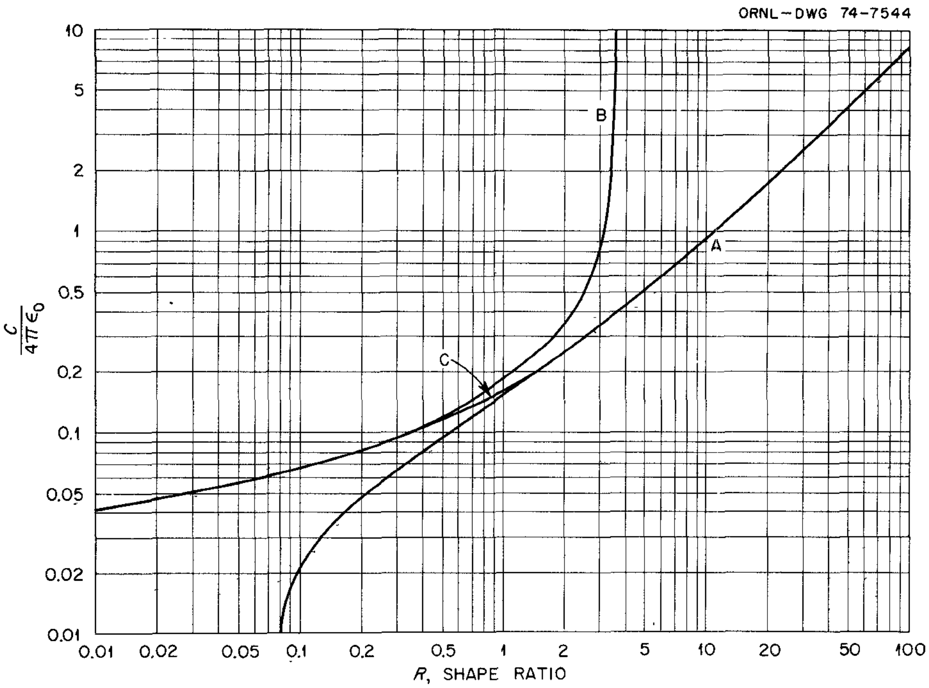

Fig. 2. Comparison of capacitance approximation formulas. Curve *A* is Bromwich's wide-line formula; *B* is the round-conductor approximation for narrow strips, and *C* is the uniform-current approximation.

COMPARISON WITH OTHER APPROXIMATIONS

It is of interest to compare the uniform-current approximation (10) with previously published approximations. One approximation for narrow lines ($R < 1$) is obtained by assuming parallel round conductors [3]. This approximation gives

$$C = (\pi\epsilon_0)[\ln (4/R)]^{-1} \text{ F/m}. \tag{11}$$

A more refined narrow-line approximation given by Wheeler [4] is

$$C = (\pi\epsilon_0)[\ln (4/R) + (1/8)R^2]^{-1} \text{ F/m}. \tag{12}$$

An early wide-line ($R > 1$) approximation due to Bromwich [5] is

$$C = (R\epsilon_0)[1 + (1/\pi R)(1 + \ln 2\pi R)] \text{ F/m} \tag{13}$$

and Wheeler's [4] wide-line formula is

$$C = (R\epsilon_0)[1 + (1/\pi R) \ln (17.08R + 15.71] \text{ F/m}. \tag{14}$$

Equations (10)–(14) are compared in Table I, which indicates that the uniform-current approximation is good for the entire range of R values. It agrees well with (11) and (12) for narrow lines and with (13) and (14) for wide lines. Equations (10), (11), and (13) are graphically compared in Fig. 2.

Comparison with a few exact values calculated from (1) indicates that the maximum inaccuracy of the uniform-current approximation occurs near $R = 1$ and is about -5 percent. This inaccuracy could probably be reduced by a suitable choice of empirical correction factors.

It is worthwhile to mention that an inexact source distribution must lead to a stored energy which is greater than the actual value. The uniform-current assumption is not exact, so the resulting energy given by (8) must be in excess of the actual stored energy. This in turn means that the inductance approximation in (9) is always greater than the exact inductance, and that the capacitance approximation in (10) is always less than the exact capacitance.

The fact that the capacitance obtained with the uniform-current approximation is reasonably good for all R values means that (8) for the stored energy must also be a reasonably good approximation for all R values. This must mean that the gradient of (8) is very close to the actual energy gradient, i.e., that the force expression in (4) is a very good approximation and should be useful in structural design calculations for thin, parallel bus bars.

All of the results given here can also be obtained by starting with a uniform charge distribution. The resulting energy expression is identical to (8) with μI^2 replaced by Q^2/ϵ.

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An Accurate Determination of the Characteristic Impedance of the Coaxial System Consisting of a Square Concentric with a Circle

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Bowman [1] has shown how the rectangular region in the w plane of Fig. 1, bounded by $0ABC$, may be mapped conformally into the trapezoidal region in the z plane of Fig. 1, bounded by $0ABC$, by means of the successive transformations

$$t = \mathrm{sn}^2\,(w,k) \tag{1}$$

$$u = \frac{k}{k'}\,(t-1)^{1/2} \tag{2}$$

$$\zeta = \left[\frac{2(k+ik')u}{(1+u)(k+ik'u)}\right]^{1/2} \tag{3}$$

$$z = \frac{1-i}{k+ik'}\left\{\int_0^\zeta \frac{d\zeta}{[(1-\zeta^2)(1-\lambda^2\zeta^2)]^{1/2}} - \int_0^\zeta \frac{d\zeta}{[(1-\zeta^2)(1-\lambda'^2\zeta^2)]^{1/2}}\right\} \tag{4}$$

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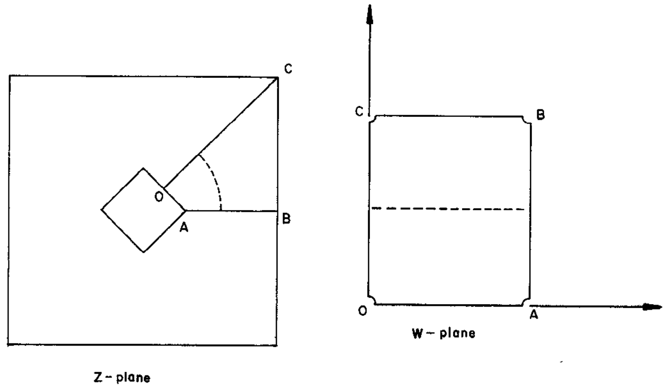

Fig. 1. Z and W coordinate planes.

where

$$\lambda = \frac{k^{1/2} + \dfrac{1+i}{(2)^{1/2}}\,(k')^{1/2}}{(2)^{1/2}(k+ik')^{1/2}}$$

and

$$\lambda' = \frac{(k)^{1/2} - \dfrac{1+i}{(2)^{1/2}}\,(k')^{1/2}}{(2)^{1/2}(k+ik')^{1/2}}.$$

Of course

$$\lambda^2 + \lambda'^2 = 1.$$

It is clear that the equipotentials in the w plane which are horizontal lines will map into portions of closed curves in the z plane which encircle the inner square and that for one of these equipotentials, points of intersection of the corresponding equipotential in the z plane with the line segments $0C$ and AB are equidistant from the center of the inner square. We are thus assured of the existence of an equipotential in the z plane which is equidistant from the center of the system at eight points spaced 45° apart. It is the object of this letter to show that this curve differs very little from a circle for the cases of greatest interest and that we have a means of determining with considerable accuracy the characteristic impedance of a coaxial line in which one of the conductors has a square cross section while the other has a circular cross section.

The problem of locating this equipotential in the z plane, and determining how much it differs from a circle depends on the evaluation of the incomplete elliptic integrals of the first kind in (4) in which the moduli are complex and the integration is performed, in general, along curves in the complex ζ plane. It turns out that the theory of second-order modular transformations [2], Landen transformations in complex form, is entirely adequate for the purpose, and that the integrals may be evaluated with the help of a digital computer with great accuracy.

Fig. 2 plots the values of the characteristic impedances for the cases where the outer conductor is a square or a circle as a function of the ratio of the circumference of the outer conductor to that of the inner conductor. For impedance values in the vicinity of 25 Ω, the curved equipotential is circular within ±0.2 percent, while at the other extreme, where the impedance values are approximately 100 Ω, the curved equipotential is circular within 2 parts in 10^7.

The reader should be reminded that k is the variable in terms of which the two characteristic impedances and the other variables of the problem may be expressed parametrically. It enters the analysis as the modulus of the Jacobian elliptic function, sn (w,k) by means of which the rectangle, $0ABC$, in the w plane is mapped into the upper-half t plane by (1). k varies from 0 to 1 and so controls the